

On the Structure of Quantum Gauge Theories with External Fields

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We consider generating functionals of Green's functions with external fields in the framework of BV and BLT quantization schemes for general gauge theories. The corresponding Ward identities are obtained, and the gauge dependence is studied.

1. Introduction

The Ward identities (i.e. relations between Green's functions resulting from initial classical invariance) are the basic means providing insight into the quantum structure of gauge theories. Thus, they are indispensable for the proof of gauge-invariant renormalizability of general gauge theories in both the BV [1] and BLT [2] quantization approaches. Also based on the use of the Ward identities was the investigation of gauge dependence, originally, in Yang-Mills theory [3] and, afterwards, in general gauge theories [4, 5] (this equally refers to theories with composite fields [6, 7]). At the same time, in quantum general gauge theories (both non-renormalized and renormalized ones), the Ward identities underlie the proof of the existence of Noether charge operators [8] with the algebraic properties required for the analysis of unitarity conditions [9].

Considerably less appreciated, however, is the effect which Ward identities may have on the study of quantum gauge theories with external fields

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(although examples of such theories, arousing interest in the corresponding Ward identities, recently have appeared: see, for instance, [10]).

The concept of external fields is basically utilized in the study of theories (notably those coupled to gravity) that are faced with difficulties in performing integration over a part of the variables in the path integral. The well-known attempts to handle such problems in quantum field theory consist in lifting functional integration from these variables (normally, a part of the initial classical fields), which are then treated as external. Note that so far this has been considered as the only reliable approach — in the absence of a consistent theory of quantum gravity — permitting to take into account such gravitational effects that pertain to the GUT epoch of the Universe, e.g. to incorporate the effects of gravity into the early Universe cosmology and to address the numerous problems arising in the physics of black holes.

In this connection, the purpose of our paper is to derive, within the BV (section 2) and BLT (section 3) quantization methods for arbitrary gauge theories, the Ward identities for the generating functionals of Green's functions with external fields and to investigate the character of their dependence on the most general form of gauge fixing.

We use the condensed notations introduced by De Witt [11] and designations adopted in [1, 2]. The invariant tensor of the group $Sp(2)$ is denoted as ε^{ab} ($a = 1, 2$). It is a constant antisymmetric second rank tensor and subject to the normalization condition $\varepsilon^{12} = 1$. The symmetrization of a certain quantity over the $Sp(2)$ indices is denoted as $A^{\{ab\}} = A^{ab} + A^{ba}$. Derivatives with respect to sources and antifields are understood as acting from the left, and those to fields, as acting from the right (unless otherwise specified); left derivatives with respect to the fields are labelled by the subscript “ l ” ($\delta_l/\delta\phi$ stands for the left derivative with respect to the field ϕ).

2. Quantum Gauge Theories with External Fields in the BV Formalism

Let us recall that the quantization of a gauge theory within the BV method [1] (see also [12]) implies introducing a complete set of fields ϕ^A and a set of the corresponding antifields ϕ_A^* (which play the role of sources of the BRST

transformations) with the following Grassmann parities:

$$\varepsilon(\phi^A) \equiv \varepsilon_A, \quad \varepsilon(\phi_A^*) = \varepsilon_A + 1.$$

The specific structure of the configuration space of the fields ϕ^A (composed by the original classical fields, the (anti) ghost pyramids and the Lagrangian multipliers) is determined by the properties of the initial classical theory, i.e. by the linear dependence (for reducible theories) or independence (for irreducible theories) of the generators of gauge transformations.

The extended generating functional $Z(J, \phi^*)$ of the Green functions for the fields of the complete configuration space is constructed within the BV quantization scheme by the rule

$$Z(J, \phi^*) = \int d\phi \exp \left\{ \frac{i}{\hbar} \left(S_{\text{ext}}(\phi, \phi^*) + J_A \phi^A \right) \right\}, \quad (1)$$

where \hbar is the Planck constant, J_A are the usual sources to the fields ϕ^A , $\varepsilon(J_A) = \varepsilon_A$, and $S_{\text{ext}} = S_{\text{ext}}(\phi, \phi^*)$ is the gauge fixed quantum action defined by the relation

$$\exp \left\{ \frac{i}{\hbar} S_{\text{ext}} \right\} = \exp \left(\hat{T}(\Psi) \right) \exp \left\{ \frac{i}{\hbar} S \right\}. \quad (2)$$

Here, $S = S(\phi, \phi^*)$ is a bosonic functional satisfying the equation

$$\frac{1}{2}(S, S) = i\hbar \Delta S, \quad (3)$$

or equivalently

$$\Delta \exp \left\{ \frac{i}{\hbar} S \right\} = 0, \quad (4)$$

with the boundary condition (\mathcal{S} is the initial gauge invariant classical action)

$$S|_{\phi^*=\hbar=0} = \mathcal{S}.$$

The operator $\hat{T}(\Psi)$ has the form

$$\hat{T}(\Psi) = [\Delta, \Psi]_+ , \quad (5)$$

where Ψ is a fermionic functional fixing a specific choice of admissible gauge. In eqs. (3)–(5) we use the standard definition of the antibracket (\cdot, \cdot) , given for two arbitrary functionals $F = F(\phi, \phi^*)$, $G = G(\phi, \phi^*)$ by the rule

$$(F, G) = \frac{\delta F}{\delta \phi^A} \frac{\delta G}{\delta \phi_A^*} - (-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)} \frac{\delta G}{\delta \phi^A} \frac{\delta F}{\delta \phi_A^*},$$

and the operator Δ is given by

$$\Delta = (-1)^{\varepsilon_A} \frac{\delta_l}{\delta \phi^A} \frac{\delta}{\delta \phi_A^*}.$$

As a consequence of the nilpotency of Δ , the relation $[\hat{T}(\Psi), \Delta] = 0$ implies that the functional S_{ext} in (2) satisfies an equation of the same form as (4)

$$\Delta \exp \left\{ \frac{i}{\hbar} S_{\text{ext}} \right\} = 0. \quad (6)$$

It is well-known that the gauge fixing (2) and (5) is in fact a particular case of the transformation generated by $\hat{T}(\Psi)$ with any fermionic operator chosen for Ψ and describing the arbitrariness of solutions of eq. (3). In this connection, we shall further deal with the most general case of gauge fixing, Ψ being an arbitrary operator-valued functional.

Let us consider the following representation of the generating functional $Z(J, \phi^*)$ in eq. (1):

$$\begin{aligned} Z(J, \phi^*) &= \int d\psi \mathcal{Z}(\mathcal{J}, \psi, \phi^*) \exp \left(\frac{i}{\hbar} \mathcal{Y}\psi \right), \\ \mathcal{Z}(\mathcal{J}, \psi, \phi^*) &= \int d\varphi \exp \left\{ \frac{i}{\hbar} \left(S_{\text{ext}}(\varphi, \psi, \phi^*) + \mathcal{J}\varphi \right) \right\} \end{aligned} \quad (7)$$

with the decomposition

$$\phi^A = (\varphi^i, \psi^\alpha), \quad J_A = (\mathcal{J}_i, \mathcal{Y}_\alpha),$$

$$\varepsilon(\varphi^i) \equiv \varepsilon_i, \quad \varepsilon(\psi^\alpha) \equiv \varepsilon_\alpha.$$

In what follows we shall refer to $\mathcal{Z}(\mathcal{J}, \psi, \phi^*)$ in eq. (7) as the extended generating functional of Green's functions with external fields ψ^α . At the same

time, given the set ϕ^A , we impose no restrictions on the structure of the subset ψ^α .

In order to derive the Ward identities for a quantum gauge theory of general kind with external fields in the BV quantization approach we shall take advantage of the relation (6) for S_{ext} . Indeed, multiplying eq. (6) from the left by $\exp\{i/\hbar \mathcal{J}\varphi\}$ and integrating over the fields φ^i , we have

$$\int d\varphi \exp\left(\frac{i}{\hbar} \mathcal{J}_i \varphi^i\right) \Delta \exp\left\{\frac{i}{\hbar} S_{\text{ext}}(\varphi, \psi, \phi^*)\right\} = 0. \quad (8)$$

Then, using the equality

$$\exp\left\{\frac{i}{\hbar} \mathcal{J}_i \varphi^i\right\} \Delta = \left(\Delta - \frac{i}{\hbar} \mathcal{J}_i \frac{\delta}{\delta \varphi_i^*}\right) \exp\left\{\frac{i}{\hbar} \mathcal{J}_i \varphi^i\right\} \quad (9)$$

and integrating in eq. (8) by parts, we obtain for $\mathcal{Z} = \mathcal{Z}(\mathcal{J}, \psi, \phi^*)$ the following Ward identity:

$$\hat{\omega} \mathcal{Z} = 0, \quad (10)$$

where $\hat{\omega}$ is the operator

$$\hat{\omega} = i\hbar \Delta_\psi + \mathcal{J}_i \frac{\delta}{\delta \varphi_i^*}, \quad \Delta_\psi \equiv (-1)^{\varepsilon_\alpha} \frac{\delta_l}{\delta \psi^\alpha} \frac{\delta}{\delta \psi_\alpha^*}, \quad (11)$$

possessing nilpotency $\hat{\omega}^2 = 0$.

One readily establishes the fact that in terms of the generating functional of the connected Green functions with external fields $\mathcal{W} = \mathcal{W}(\mathcal{J}, \psi, \phi^*)$ defined by

$$\mathcal{Z} = \exp\left\{\frac{i}{\hbar} \mathcal{W}\right\}$$

the identity (10) takes on the form

$$\hat{\omega} \mathcal{W} = \frac{\delta \mathcal{W}}{\delta \psi^\alpha} \frac{\delta \mathcal{W}}{\delta \psi_\alpha^*}. \quad (12)$$

Similarly, introducing the generating functional $\Gamma = \Gamma(\varphi, \psi, \phi^*)$ of the vertex Green functions with external fields

$$\Gamma(\varphi, \psi, \phi^*) = \mathcal{W}(\mathcal{J}, \psi, \phi^*) - \mathcal{J}_i \varphi^i, \quad \varphi^i = \frac{\delta \mathcal{W}}{\delta \mathcal{J}_i}, \quad \mathcal{J}_i = -\frac{\delta \Gamma}{\delta \varphi^i},$$

we have, by virtue of eqs. (11), (12) and the relations

$$\frac{\delta \mathcal{W}}{\delta \psi^\alpha} = \frac{\delta \Gamma}{\delta \psi^\alpha}, \quad \frac{\delta \mathcal{W}}{\delta \phi_A^*} = \frac{\delta \Gamma}{\delta \phi_A^*},$$

the following Ward identities:

$$\frac{1}{2}(\Gamma, \Gamma) = i\hbar \Delta_\psi \Gamma - i\hbar (\Gamma''^{-1})^{ij} \left(\frac{\delta_l}{\delta \varphi^j} \frac{\delta \Gamma}{\delta \psi^\alpha} \right) \left(\frac{\delta}{\delta \psi_\alpha^*} \frac{\delta \Gamma}{\delta \varphi^i} \right), \quad (13)$$

where

$$(\Gamma''^{-1})^{ik} (\Gamma'')_{kj} = \delta_j^i, \quad (\Gamma'')_{ij} \equiv \frac{\delta_l}{\delta \varphi^i} \frac{\delta \Gamma}{\delta \varphi^j}.$$

Let us now investigate the gauge dependence of the above-introduced generating functionals with external fields under the most general variation

$$\delta \Psi \left(\phi^A, \phi_A^*; \frac{\delta_l}{\delta \phi^A}, \frac{\delta}{\delta \phi_A^*} \right) = \delta \Psi \left(\varphi^i, \psi^\alpha, \phi_A^*; \frac{\delta_l}{\delta \varphi^i}, \frac{\delta_l}{\delta \psi^\alpha}, \frac{\delta}{\delta \phi_A^*} \right)$$

of the gauge fermion.

From eq. (2) it follows that the variation of $\exp\{i/\hbar S_{\text{ext}}\}$ reads

$$\delta \left(\exp \left\{ \frac{i}{\hbar} S_{\text{ext}} \right\} \right) = \hat{T}(\delta X) \exp \left\{ \frac{i}{\hbar} S_{\text{ext}} \right\},$$

where δX is related to $\delta \Psi$ through an operator-valued transformation (linear in $\delta \Psi$) whose explicit form is not essential for the following treatment. Note, however, that we can always dispose of the operator ordering so that δX is represented as

$$\begin{aligned} \delta X \left(\varphi^i, \psi^\alpha, \phi_A^*; \frac{\delta_l}{\delta \varphi^i}, \frac{\delta_l}{\delta \psi^\alpha}, \frac{\delta}{\delta \phi_A^*} \right) &= \delta X^{(0)} \left(\varphi^i, \psi^\alpha, \phi_A^*; \frac{\delta_l}{\delta \psi^\alpha}, \frac{\delta}{\delta \phi_A^*} \right) \\ &+ \sum_{N=1} \frac{\delta_l}{\delta \varphi^{i_1}} \cdots \frac{\delta_l}{\delta \varphi^{i_N}} \delta X^{(i_1 \dots i_N)} \left(\varphi^i, \psi^\alpha, \phi_A^*; \frac{\delta_l}{\delta \psi^\alpha}, \frac{\delta}{\delta \phi_A^*} \right). \end{aligned} \quad (14)$$

By virtue of eqs. (5)–(7), the variation of the functional $\mathcal{Z}(\mathcal{J}, \psi, \phi^*)$ has the form

$$\delta \mathcal{Z}(\mathcal{J}, \psi, \phi^*) = \int d\varphi \exp \left(\frac{i}{\hbar} \mathcal{J}_i \varphi^i \right) \Delta \left(\delta X \exp \left\{ \frac{i}{\hbar} S_{\text{ext}}(\varphi, \psi, \phi^*) \right\} \right). \quad (15)$$

Then, using eq. (9), definition (11) and the representation (14), we have, after the integration by parts in eq. (15) has been performed,

$$\delta \mathcal{Z}(\mathcal{J}, \psi, \phi^*) = \frac{1}{i\hbar} \hat{\omega} \delta X \left(\frac{\hbar}{i} \frac{\delta}{\delta \mathcal{J}_i}, \psi^\alpha, \phi_A^*; (-1)^{\varepsilon_i} \frac{1}{i\hbar} \mathcal{J}_i, \frac{\delta_l}{\delta \psi^\alpha}, \frac{\delta}{\delta \phi_A^*} \right) \mathcal{Z}(\mathcal{J}, \psi, \phi^*). \quad (16)$$

One readily observes that in terms of the generating functional $\mathcal{W}(\mathcal{J}, \psi, \phi^*)$ the relation (16) takes on the form

$$\delta \mathcal{W} = -\hat{\Omega} \langle \delta X \rangle, \quad (17)$$

where $\langle \delta X \rangle$ is the vacuum expectation value of the functional δX

$$\langle \delta X \rangle = \delta X \left(\frac{\delta \mathcal{W}}{\delta \mathcal{J}_i} + \frac{\hbar}{i} \frac{\delta}{\delta \mathcal{J}_i}, \psi^\alpha, \phi_A^*; (-1)^{\varepsilon_i} \frac{1}{i\hbar} \mathcal{J}_i, \frac{i}{\hbar} \frac{\delta_l \mathcal{W}}{\delta \psi^\alpha} + \frac{\delta_l}{\delta \psi^\alpha}, \frac{i}{\hbar} \frac{\delta \mathcal{W}}{\delta \phi_A^*} + \frac{\delta}{\delta \phi_A^*} \right),$$

and $\hat{\Omega}$ is an operator defined according to

$$\hat{\Omega} = \exp \left\{ -\frac{i}{\hbar} \mathcal{W} \right\} \hat{\omega} \exp \left\{ \frac{i}{\hbar} \mathcal{W} \right\}.$$

By virtue of the Ward identities (12) for the functional $\mathcal{W}(\mathcal{J}, \psi, \phi^*)$, the operator $\hat{\Omega}$ can be represented as

$$\hat{\Omega} = \hat{\omega} - \frac{\delta \mathcal{W}}{\delta \psi^\alpha} \frac{\delta}{\delta \psi_\alpha^*} - (-1)^{\varepsilon_\alpha} \frac{\delta \mathcal{W}}{\delta \psi_\alpha^*} \frac{\delta_l}{\delta \psi^\alpha}, \quad (18)$$

while from the nilpotency of $\hat{\omega}$ it follows that

$$\hat{\Omega}^2 = 0. \quad (19)$$

In order to determine the character of the gauge dependence of $\Gamma(\varphi, \psi, \phi^*)$ we first observe that $\delta \Gamma = \delta \mathcal{W}$ and, secondly,

$$\begin{aligned} \left. \frac{\delta}{\delta \mathcal{J}_i} \right|_{\psi, \phi^*} &= \left. \frac{\delta \varphi^j}{\delta \mathcal{J}_i} \frac{\delta_l}{\delta \varphi^j} \right|_{\psi, \phi^*}, \\ \left. \frac{\delta_l}{\delta \psi^\alpha} \right|_{\mathcal{J}, \phi^*} &= \left. \frac{\delta_l}{\delta \psi^\alpha} \right|_{\varphi, \phi^*} + \left. \frac{\delta_l \varphi^i}{\delta \psi^\alpha} \frac{\delta_l}{\delta \varphi^i} \right|_{\psi, \phi^*}, \\ \left. \frac{\delta}{\delta \phi_A^*} \right|_{\mathcal{J}, \psi} &= \left. \frac{\delta}{\delta \phi_A^*} \right|_{\varphi, \psi} + \left. \frac{\delta \varphi^i}{\delta \phi_A^*} \frac{\delta_l}{\delta \varphi^i} \right|_{\psi, \phi^*}. \end{aligned}$$

At the same time, note that

$$\begin{aligned}\frac{\delta\varphi^j}{\delta\mathcal{J}_i} &= -(\Gamma''^{-1})^{ij}, \\ \frac{\delta_l\varphi^i}{\delta\psi^\alpha} &= -(-1)^{\varepsilon_i\varepsilon_\alpha}(\Gamma''^{-1})^{ij}\left(\frac{\delta_l}{\delta\varphi^j}\frac{\delta_l\Gamma}{\delta\psi^\alpha}\right), \\ \frac{\delta\varphi^i}{\delta\phi_A^*} &= -(-1)^{\varepsilon_i(\varepsilon_A+1)}(\Gamma''^{-1})^{ij}\left(\frac{\delta_l}{\delta\varphi^j}\frac{\delta\Gamma}{\delta\phi_A^*}\right),\end{aligned}$$

where

$$\frac{\delta_l\mathcal{J}_j}{\delta\varphi^i} = -(\Gamma'')_{ij}.$$

By virtue of eq. (17) and the above equations, the variation of $\Gamma(\varphi, \psi, \phi^*)$ reads

$$\delta\Gamma = -\hat{q}\langle\langle\delta X\rangle\rangle. \quad (20)$$

Here,

$$\begin{aligned}\langle\langle\delta X\rangle\rangle &= \delta X\left(\varphi^i + i\hbar(\Gamma''^{-1})^{ij}\frac{\delta_l}{\delta\varphi^j}, \psi^\alpha, \phi_A^*; \frac{i}{\hbar}\frac{\delta_l\Gamma}{\delta\varphi^i}, \right. \\ &\quad \left. \frac{\delta_l}{\delta\psi^\alpha} + \frac{i}{\hbar}\frac{\delta_l\Gamma}{\delta\psi^\alpha} - (-1)^{\varepsilon_i\varepsilon_\alpha} \cdot (\Gamma''^{-1})^{ij}\left(\frac{\delta_l}{\delta\varphi^j}\frac{\delta_l\Gamma}{\delta\psi^\alpha}\right)\frac{\delta_l}{\delta\varphi^i}, \right. \\ &\quad \left. \frac{\delta}{\delta\phi_A^*} + \frac{i}{\hbar}\frac{\delta\Gamma}{\delta\phi_A^*} - (-1)^{\varepsilon_i(\varepsilon_A+1)}(\Gamma''^{-1})^{ij}\left(\frac{\delta_l}{\delta\varphi^j}\frac{\delta\Gamma}{\delta\phi_A^*}\right)\frac{\delta_l}{\delta\varphi^i}\right),\end{aligned}$$

and \hat{q} is an operator of the form

$$\begin{aligned}\hat{q} &= i\hbar(-1)^{\varepsilon_\alpha}\left(\frac{\delta_l}{\delta\psi^\alpha} - (-1)^{\varepsilon_i\varepsilon_\alpha} \cdot (\Gamma''^{-1})^{ij}\left(\frac{\delta_l}{\delta\varphi^j}\frac{\delta_l\Gamma}{\delta\psi^\alpha}\right)\frac{\delta_l}{\delta\varphi^i}\right) \\ &\quad \cdot \left(\frac{\delta}{\delta\psi_\alpha^*} - (-1)^{\varepsilon_m(\varepsilon_\alpha+1)}(\Gamma''^{-1})^{mn}\left(\frac{\delta_l}{\delta\varphi^n}\frac{\delta\Gamma}{\delta\psi_\alpha^*}\right)\frac{\delta_l}{\delta\varphi^m}\right) \\ &\quad - \frac{\delta\Gamma}{\delta\phi_A^*}\left(\frac{\delta}{\delta\phi_A^*} - (-1)^{\varepsilon_i(\varepsilon_A+1)}(\Gamma''^{-1})^{ij}\left(\frac{\delta_l}{\delta\varphi^j}\frac{\delta\Gamma}{\delta\phi_A^*}\right)\frac{\delta_l}{\delta\varphi^i}\right) \\ &\quad - (-1)^{\varepsilon_\alpha}\frac{\delta\Gamma}{\delta\psi_\alpha^*}\left(\frac{\delta_l}{\delta\psi^\alpha} - (-1)^{\varepsilon_i\varepsilon_\alpha} \cdot (\Gamma''^{-1})^{ij}\left(\frac{\delta_l}{\delta\varphi^j}\frac{\delta_l\Gamma}{\delta\psi^\alpha}\right)\frac{\delta_l}{\delta\varphi^i}\right).\end{aligned} \quad (21)$$

Since \hat{q} is related to $\hat{\Omega}$ in (18) through the Legendre transformation, it also possesses the property of nilpotency (cf. eq. (19))

$$\hat{q}^2 = 0.$$

3. Theories with External Fields in the BLT Scheme

We now consider the structure of quantum gauge theories with external fields within the BLT quantization method [2]. To this end, we remind that the variables of the formalism in [2] are composed by fields ϕ^A and a set of the corresponding antifields ϕ_{Aa}^* , $\bar{\phi}_A$ (the doublets of antifields ϕ_{Aa}^* play the role of sources of the BRST and antiBRST transformations, while the antifields $\bar{\phi}_A$ are the sources of the mixed BRST and antiBRST transformations) with

$$\varepsilon(\phi^A) = \varepsilon_A, \quad \varepsilon(\phi_{Aa}^*) = \varepsilon_A + 1, \quad \varepsilon(\bar{\phi}_A) = \varepsilon_A.$$

Note that for any given gauge theory the structure of the configuration space of the fields ϕ^A in the BLT formalism coincides with that of the BV method. At the same time, the fields in the BLT scheme form components of irreducible completely symmetric $Sp(2)$ -tensors.

The extended generating functional $Z(J, \phi^*, \bar{\phi})$ of Green's functions for the fields of the complete configuration space is defined in the quantization scheme [2] by the rule

$$Z(J, \phi^*, \bar{\phi}) = \int d\phi \exp \left\{ \frac{i}{\hbar} \left(S_{\text{ext}}(\phi, \phi^*, \bar{\phi}) + J_A \phi^A \right) \right\}, \quad (22)$$

where $S_{\text{ext}} = S_{\text{ext}}(\phi, \phi^*, \bar{\phi})$ is the gauge fixed quantum action

$$\exp \left\{ \frac{i}{\hbar} S_{\text{ext}} \right\} = \exp \left(-i\hbar \hat{T}(F) \right) \exp \left\{ \frac{i}{\hbar} S \right\}. \quad (23)$$

Here, $S = S(\phi, \phi^*, \bar{\phi})$ is a bosonic functional satisfying the equations

$$\frac{1}{2}(S, S)^a + V^a S = i\hbar \Delta^a S, \quad (24)$$

or equivalently

$$\bar{\Delta}^a \exp \left\{ \frac{i}{\hbar} S \right\} = 0, \quad \bar{\Delta}^a \equiv \Delta^a + \frac{i}{\hbar} V^a, \quad (25)$$

with the boundary condition (\mathcal{S} is the classical action)

$$S|_{\phi^*=\bar{\phi}=\hbar=0} = \mathcal{S}.$$

$\hat{T}(F)$ is an operator defined by

$$\hat{T}(F) = \frac{1}{2}\varepsilon_{ab}[\bar{\Delta}^b, [\bar{\Delta}^a, F]_-]_+, \quad (26)$$

where F is a bosonic (generally, operator-valued) functional fixing a specific choice of gauge. In eqs. (24)–(26) we use the definition [2] of the extended antibrackets $(\cdot, \cdot)^a$, introduced for two arbitrary functionals $F = F(\phi, \phi^*, \bar{\phi})$ and $G = G(\phi, \phi^*, \bar{\phi})$ by the rule

$$(F, G)^a = \frac{\delta F}{\delta \phi^A} \frac{\delta G}{\delta \phi_{Aa}^*} - (-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)} \frac{\delta G}{\delta \phi^A} \frac{\delta F}{\delta \phi_{Aa}^*},$$

and the operators Δ^a and V^a

$$\Delta^a = (-1)^{\varepsilon_A} \frac{\delta_l}{\delta \phi^A} \frac{\delta}{\delta \phi_{Aa}^*}, \quad V^a = \varepsilon^{ab} \phi_{Ab}^* \frac{\delta}{\delta \bar{\phi}_A}.$$

As a consequence of the algebraic properties of $\bar{\Delta}^a$ in (25), i.e. $\bar{\Delta}^{\{a} \bar{\Delta}^{b\}} = 0$, the relations $[\hat{T}(F), \bar{\Delta}^a] = 0$ imply that the functional S_{ext} in (23) satisfies the equations

$$\bar{\Delta}^a \exp \left\{ \frac{i}{\hbar} S_{\text{ext}} \right\} = 0. \quad (27)$$

Let us now introduce the following representation of the generating functional $Z(\phi, \phi^*, \bar{\phi})$ in eq. (22):

$$Z(J, \phi^*, \bar{\phi}) = \int d\psi \mathcal{Z}(\mathcal{J}, \psi, \phi^*, \bar{\phi}) \exp \left(\frac{i}{\hbar} \mathcal{Y}\psi \right),$$

where $\mathcal{Z}(\mathcal{J}, \psi, \phi^*, \bar{\phi})$ denotes the extended generating functional of Green's functions with the external fields ψ^α

$$\mathcal{Z}(\mathcal{J}, \psi, \phi^*, \bar{\phi}) = \int d\varphi \exp \left\{ \frac{i}{\hbar} \left(S_{\text{ext}}(\varphi, \psi, \phi^*, \bar{\phi}) + \mathcal{J}\varphi \right) \right\}. \quad (28)$$

Here, as before, we abbreviate $\phi^A = (\varphi^i, \psi^\alpha)$ and $J_A = (\mathcal{J}_i, \mathcal{Y}_\alpha)$.

As in the case of the BV quantization scheme, the Ward identities for a quantum gauge theory of general kind with external fields considered in

the BLT method follow immediately from the equation for the gauge fixed quantum action S_{ext} . Eq. (27) implies

$$\int d\varphi \exp\left(\frac{i}{\hbar}\mathcal{J}_i\varphi^i\right)\bar{\Delta}^a \exp\left\{\frac{i}{\hbar}S_{\text{ext}}(\varphi, \psi, \phi^*, \bar{\phi})\right\} = 0. \quad (29)$$

Then, taking into account the equalities

$$\exp\left\{\frac{i}{\hbar}\mathcal{J}_i\varphi^i\right\}\bar{\Delta}^a = \left(\bar{\Delta}^a - \frac{i}{\hbar}\mathcal{J}_i\frac{\delta}{\delta\varphi_{ia}^*}\right)\exp\left\{\frac{i}{\hbar}\mathcal{J}_i\varphi^i\right\} \quad (30)$$

and performing the integration by parts in eq. (29), we have

$$\hat{\omega}^a \mathcal{Z} = 0, \quad (31)$$

where $\hat{\omega}^a$ stand for the operators

$$\hat{\omega}^a = i\hbar\Delta_\psi^a + \mathcal{J}_i\frac{\delta}{\delta\varphi_{ia}^*} - V^a, \quad \Delta_\psi^a \equiv (-1)^{\varepsilon_\alpha}\frac{\delta_l}{\delta\psi^\alpha}\frac{\delta}{\delta\psi_{\alpha a}^*} \quad (32)$$

with the properties

$$\hat{\omega}^{\{a}\hat{\omega}^{b\}} = 0. \quad (33)$$

The relations (31) together with (32) determine the Ward identities for the generating functional $\mathcal{Z}(\mathcal{J}, \psi, \phi^*, \bar{\phi})$ in (28).

Rewriting eqs. (31) in terms of the generating functionals \mathcal{W} , with $\mathcal{Z} = \exp\{i/\hbar\mathcal{W}\}$, and Γ , where

$$\Gamma(\varphi, \psi, \phi^*, \bar{\phi}) = \mathcal{W}(\mathcal{J}, \psi, \phi^*, \bar{\phi}) - \mathcal{J}_i\varphi^i, \quad \varphi^i = \frac{\delta\mathcal{W}}{\delta\mathcal{J}_i}, \quad \mathcal{J}_i = -\frac{\delta\Gamma}{\delta\varphi^i},$$

$$\frac{\delta\mathcal{W}}{\delta\psi^\alpha} = \frac{\delta\Gamma}{\delta\psi^\alpha}, \quad \frac{\delta\mathcal{W}}{\delta\phi_{Aa}^*} = \frac{\delta\Gamma}{\delta\phi_{Aa}^*}, \quad \frac{\delta\mathcal{W}}{\delta\bar{\phi}_A} = \frac{\delta\Gamma}{\delta\bar{\phi}_A},$$

we recast the Ward identities into the form

$$\hat{\omega}^a \mathcal{W} = \frac{\delta\mathcal{W}}{\delta\psi^\alpha}\frac{\delta\mathcal{W}}{\delta\psi_{\alpha a}^*}, \quad (34)$$

$$\frac{1}{2}(\Gamma, \Gamma)^a + V^a\Gamma = i\hbar\Delta_\psi^a\Gamma - i\hbar(\Gamma''^{-1})^{ij}\left(\frac{\delta_l}{\delta\varphi^j}\frac{\delta\Gamma}{\delta\psi^\alpha}\right)\left(\frac{\delta}{\delta\psi_{\alpha a}^*}\frac{\delta\Gamma}{\delta\varphi^i}\right). \quad (35)$$

We shall now study the change of the above-introduced generating functionals \mathcal{Z} , \mathcal{W} , Γ in the BLT quantization scheme under the variation of the gauge boson F , chosen in the most general form of an operator-valued functional, i.e.

$$\delta F\left(\phi^A, \phi_{Aa}^*, \bar{\phi}_A; \frac{\delta_l}{\delta \phi^A}, \frac{\delta}{\delta \phi_{Aa}^*}, \frac{\delta}{\delta \bar{\phi}_A}\right) = \delta F\left(\varphi^i, \psi^\alpha, \phi_{Aa}^*, \bar{\phi}_A; \frac{\delta_l}{\delta \varphi^i}, \frac{\delta_l}{\delta \psi^\alpha}, \frac{\delta}{\delta \phi_{Aa}^*}, \frac{\delta}{\delta \bar{\phi}_A}\right).$$

Taking eq. (23) into account, we have

$$\delta\left(\exp\left\{\frac{i}{\hbar}S_{\text{ext}}\right\}\right) = -i\hbar\hat{T}(\delta Y)\exp\left\{\frac{i}{\hbar}S_{\text{ext}}\right\}$$

and consequently, with eqs. (25) and (26),

$$\delta\mathcal{Z}(\mathcal{J}, \psi, \phi^*, \bar{\phi}) = \frac{i\hbar}{2}\varepsilon_{ab}\int d\varphi \exp\left(\frac{i}{\hbar}\mathcal{J}_i\varphi^i\right)\bar{\Delta}^a\bar{\Delta}^b\left(\delta Y \exp\left\{\frac{i}{\hbar}S_{\text{ext}}(\varphi, \psi, \phi^*, \bar{\phi})\right\}\right), \quad (36)$$

where δY (related to δF through a linear transformation) can always be represented as

$$\begin{aligned} \delta Y\left(\varphi^i, \psi^\alpha, \phi_{Aa}^*, \bar{\phi}_A; \frac{\delta_l}{\delta \varphi^i}, \frac{\delta_l}{\delta \psi^\alpha}, \frac{\delta}{\delta \phi_{Aa}^*}, \frac{\delta}{\delta \bar{\phi}_A}\right) = \\ \delta Y^{(0)}\left(\varphi^i, \psi^\alpha, \phi_{Aa}^*, \bar{\phi}_A; \frac{\delta_l}{\delta \psi^\alpha}, \frac{\delta}{\delta \phi_{Aa}^*}, \frac{\delta}{\delta \bar{\phi}_A}\right) \\ + \sum_{N=1} \frac{\delta_l}{\delta \varphi^{i_1}} \dots \frac{\delta_l}{\delta \varphi^{i_N}} \delta Y^{(i_1 \dots i_N)}\left(\varphi^i, \psi^\alpha, \phi_{Aa}^*, \bar{\phi}_A; \frac{\delta_l}{\delta \psi^\alpha}, \frac{\delta}{\delta \phi_{Aa}^*}, \frac{\delta}{\delta \bar{\phi}_A}\right). \end{aligned} \quad (37)$$

By virtue of eqs. (30), (32), (37) and integrating by parts in eq. (36) one finds

$$\begin{aligned} \delta\mathcal{Z}(\mathcal{J}, \psi, \phi^*, \bar{\phi}) &= \frac{i}{2\hbar}\varepsilon_{ab}\hat{\omega}^b\hat{\omega}^a. \\ \cdot \delta Y\left(\frac{\hbar}{i}\frac{\delta}{\delta \mathcal{J}_i}, \psi^\alpha, \phi_{Aa}^*, \bar{\phi}_A; (-1)^{\varepsilon_i}\frac{1}{i\hbar}\mathcal{J}_i, \frac{\delta_l}{\delta \psi^\alpha}, \frac{\delta_l}{\delta \phi_{Aa}^*}, \frac{\delta_l}{\delta \bar{\phi}_A}\right) &\mathcal{Z}(\mathcal{J}, \psi, \phi^*, \bar{\phi}). \end{aligned} \quad (38)$$

Given this and using arguments quite similar to those presented in the case of the BV scheme, one readily establishes the fact that in terms of the generating functionals $\mathcal{W}(\mathcal{J}, \psi, \phi^*, \bar{\phi})$ and $\Gamma(\varphi, \psi, \phi^*, \bar{\phi})$ the corresponding variations have the form

$$\delta\mathcal{W} = \frac{1}{2}\varepsilon_{ab}\hat{\Omega}^b\hat{\Omega}^a\langle\delta Y\rangle, \quad (39)$$

$$\delta\Gamma = \frac{1}{2}\varepsilon_{ab}\hat{q}^b\hat{q}^a\langle\langle\delta Y\rangle\rangle, \quad (40)$$

where

$$\begin{aligned} \langle\delta Y\rangle = \delta Y & \left(\frac{\delta\mathcal{W}}{\delta\mathcal{J}_i} + \frac{\hbar}{i} \frac{\delta}{\delta\mathcal{J}_i}, \psi^\alpha, \phi_{Aa}^*, \bar{\phi}_A; (-1)^{\varepsilon_i} \frac{1}{i\hbar} \mathcal{J}_i, \right. \\ & \left. \frac{i}{\hbar} \frac{\delta_l \mathcal{W}}{\delta\psi^\alpha} + \frac{\delta_l}{\delta\psi^\alpha}, \frac{i}{\hbar} \frac{\delta\mathcal{W}}{\delta\phi_{Aa}^*} + \frac{\delta}{\delta\phi_{Aa}^*}, \frac{i}{\hbar} \frac{\delta\mathcal{W}}{\delta\bar{\phi}_A} + \frac{\delta}{\delta\bar{\phi}_A} \right) \end{aligned}$$

and

$$\begin{aligned} \langle\langle\delta Y\rangle\rangle = \delta Y & \left(\varphi^i + i\hbar(\Gamma''^{-1})^{ij} \frac{\delta_l}{\delta\varphi^j}, \psi^\alpha, \phi_{Aa}^*, \bar{\phi}_A; \frac{i}{\hbar} \frac{\delta_l \Gamma}{\delta\varphi^i}, \right. \\ & \frac{\delta_l}{\delta\psi^\alpha} + \frac{i}{\hbar} \frac{\delta_l \Gamma}{\delta\psi^\alpha} - (-1)^{\varepsilon_i \varepsilon_\alpha} \cdot (\Gamma''^{-1})^{ij} \left(\frac{\delta_l}{\delta\varphi^j} \frac{\delta_l \Gamma}{\delta\psi^\alpha} \right) \frac{\delta_l}{\delta\varphi^i}, \\ & \frac{\delta}{\delta\phi_{Aa}^*} + \frac{i}{\hbar} \frac{\delta\Gamma}{\delta\phi_{Aa}^*} - (-1)^{\varepsilon_i(\varepsilon_A+1)} (\Gamma''^{-1})^{ij} \left(\frac{\delta_l}{\delta\varphi^j} \frac{\delta\Gamma}{\delta\phi_{Aa}^*} \right) \frac{\delta_l}{\delta\varphi^i}, \\ & \left. \frac{\delta}{\delta\bar{\phi}_A} + \frac{i}{\hbar} \frac{\delta\Gamma}{\delta\bar{\phi}_A} - (-1)^{\varepsilon_i \varepsilon_A} \cdot (\Gamma''^{-1})^{ij} \left(\frac{\delta_l}{\delta\varphi^j} \frac{\delta\Gamma}{\delta\bar{\phi}_A} \right) \frac{\delta_l}{\delta\varphi^i} \right). \end{aligned}$$

In eq. (39), $\hat{\Omega}^a$ stand for the operators

$$\hat{\Omega}^a = \exp \left\{ -\frac{i}{\hbar} \mathcal{W} \right\} \hat{\omega}^a \exp \left\{ \frac{i}{\hbar} \mathcal{W} \right\},$$

whose explicit form is

$$\hat{\Omega}^a = \hat{\omega}^a - \frac{\delta\mathcal{W}}{\delta\psi^\alpha} \frac{\delta}{\delta\psi_{\alpha a}^*} - (-1)^{\varepsilon_\alpha} \frac{\delta\mathcal{W}}{\delta\psi_{\alpha a}^*} \frac{\delta_l}{\delta\psi^\alpha}. \quad (41)$$

Taking into account eq. (33) and the Ward identities (34) for $\mathcal{W}(\mathcal{J}, \psi, \phi^*, \bar{\phi})$, we have

$$\hat{\Omega}^{\{a} \hat{\Omega}^{b\}} = 0. \quad (42)$$

At the same time, \hat{q}^a in eq. (40) are the operators

$$\hat{q}^a = i\hbar(-1)^{\varepsilon_\alpha} \left(\frac{\delta_l}{\delta\psi^\alpha} - (-1)^{\varepsilon_i \varepsilon_\alpha} \cdot (\Gamma''^{-1})^{ij} \left(\frac{\delta_l}{\delta\varphi^j} \frac{\delta_l \Gamma}{\delta\psi^\alpha} \right) \frac{\delta_l}{\delta\varphi^i} \right).$$

$$\begin{aligned}
& \cdot \left(\frac{\delta}{\delta \psi_{\alpha a}^*} - (-1)^{\varepsilon_m(\varepsilon_\alpha+1)} (\Gamma''^{-1})^{mn} \left(\frac{\delta_l}{\delta \varphi^n} \frac{\delta \Gamma}{\delta \psi_{\alpha a}^*} \right) \frac{\delta_l}{\delta \varphi^m} \right) \\
& - \frac{\delta \Gamma}{\delta \phi^A} \left(\frac{\delta}{\delta \phi_{Aa}^*} - (-1)^{\varepsilon_i(\varepsilon_A+1)} (\Gamma''^{-1})^{ij} \left(\frac{\delta_l}{\delta \varphi^j} \frac{\delta \Gamma}{\delta \phi_{Aa}^*} \right) \frac{\delta_l}{\delta \varphi^i} \right) \quad (43) \\
& - (-1)^{\varepsilon_\alpha} \frac{\delta \Gamma}{\delta \psi_{\alpha a}^*} \left(\frac{\delta_l}{\delta \psi^\alpha} - (-1)^{\varepsilon_i \varepsilon_\alpha} (\Gamma''^{-1})^{ij} \left(\frac{\delta_l}{\delta \varphi^j} \frac{\delta \Gamma}{\delta \psi^\alpha} \right) \frac{\delta_l}{\delta \varphi^i} \right) \\
& - \varepsilon^{ab} \phi_{Ab}^* \left(\frac{\delta}{\delta \bar{\phi}_A} - (-1)^{\varepsilon_i \varepsilon_A} (\Gamma''^{-1})^{ij} \left(\frac{\delta_l}{\delta \varphi^j} \frac{\delta \Gamma}{\delta \bar{\phi}_A} \right) \frac{\delta_l}{\delta \varphi^i} \right)
\end{aligned}$$

related to $\hat{\Omega}^a$ through the Legendre transformation and, consequently, also possessing the properties (cf. eq. (42))

$$\hat{q}^{\{a} \hat{q}^{b\}} = 0.$$

4. Conclusion

In this paper we have considered generating functionals of Green's functions with external fields in the framework of BV [1] and BLT [2] quantization schemes for general gauge theories. Note that, as compared to the study of [13], our approach closely follows the standard prescriptions of [1, 2] without having recourse to introducing an additional composite operator to the path integral.

The Ward identities for the generating functionals with external fields are obtained: (10), (12), (13), (31), (34) and (35). The explicit gauge dependence on the most general form of gauge fixing has been derived: (16), (17), (20) and (38)–(40). In this connection, it is a quite remarkable fact that the gauge dependence concerned is described with the help of nilpotent fermionic operators (11), (18) and (21) in the BV formalism and doublets (32), (41) and (43) of nilpotent anticommuting operators in the BLT method, as in the case of quantum gauge theories of general kind with composite fields (see [7]).

The use of external or background fields plays an important role if a (background field) gauge invariant effective action is to be considered. A recently discussed approach is the method of the effective average action [14] which allows to derive renormalization group equations for the coupling constants of a theory in a way which goes behind the standard one-loop

calculation. However, by the present work, the way for using such methods in theories requiring BV or BLT quantization, e. g. string theories, is not yet prepared. Namely, if a general gauge theory has to be considered where the fields are to be decomposed into a quantized part φ^A and a classical background configuration ϕ_{cl}^A , i.e. $\Phi^A = \varphi^A + \phi_{\text{cl}}^A$, the present method has to be generalized. Of course, such an approach will be of interest if vacuum condensates or topological nontrivial configurations like instantons — as in the case of QCD — play an essential role. This question has been considered independently in a recent paper [15]. The extension of that method to the BV and BLT approach will be considered later on.

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